ON THE ALGEBRAIC COMPONENTS OF THE SL$(\mathbf{2}, \mathbb{C})$ CHARACTER VARIETIES OF KNOT EXTERIORS

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Abstract

We show that if a knot exterior satisfies certain conditions, then it has finite cyclic coverings with arbitrarily large numbers of nontrivial algebraic components in their $SL_2(\mathbb{C})$-character varieties (Theorem A). As an example, these conditions hold for hyperbolic punctured torus bundles over the circle (Theorem B). We investigate in more detail the finite cyclic covers of the figure-eight knot exterior and show that for every integer m there exists a finite covering such that its $SL_2(\mathbb{C})$-character variety contains curve components which have associated boundary slopes whose distance is larger than m (Theorem C). Lastly, we show that given an integer m, there exists a hyperbolic knot exterior in the 3-sphere that has a finite cyclic covering such that its $SL_2(\mathbb{C})$-character variety contains more than m norm curve components each of which contains the character of a discrete faithful presentation of the fundamental group of the covering space (Theorem D).

1 Introduction

Throughout this paper M will denote a *knot exterior*, that is a compact, connected, orientable, irreducible, boundary-irreducible 3-manifold with boundary a torus. We shall call M *hyperbolic* if its interior admits a complete Riemannian metric of finite volume and constant negative sectional curvature. A knot exterior is said to be *small* if it does not contain any closed, embedded, orientable surfaces which are *essential*, i.e. incompressible and non-boundary-parallel. It is a consequence of Thurston's *uniformisation theorem* [26] and the *torus theorem* [17] that a small knot exterior is either hyperbolic or admits a Seifert fibred structure whose orbit manifold is a disk and which has two exceptional fibres.

For any finitely generated group Γ, we use $R(\Gamma)$ to denote the set of representations of Γ into $SL(2,\mathbb{C})$ [12]. This set can be regarded as a complex affine algebraic variety (in this paper a variety may be reducible). The *character* of an element $\rho \in R(\Gamma)$ is the function

$$\chi_\rho : \Gamma \to \mathbb{C}, \qquad \chi_\rho(\gamma) = trace(\rho(\gamma)).$$

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The character of an irreducible (resp. reducible) representation is called an *irreducible* (resp. *reducible*) character. If two representations are conjugate to each other, they have the same character. The set of the characters of the representations in $R(\Gamma)$, denoted $X(\Gamma)$, is also a complex affine variety, called the $SL(2,\mathbb{C})$-*character variety* of Γ [12]. The natural surjective map $t : R(\Gamma) \to X(\Gamma)$ which sends a representation to its character is regular. When Γ is the fundamental group of space W, we shall denote $R(\Gamma)$ by $R(W)$ and $X(\Gamma)$ by $X(W)$.

Recent study has shown that $X(M)$ contains a great deal of topological information about a knot exterior M. In particular, it contains information about the manifolds obtained by Dehn filling M along ∂M [26], [12], [21], [7], [2, 3, 4]. In this paper we study $X(M)$ as an algebraic variety. More specifically, we examine how many and what kind of algebraic components $X(M)$ can have. Of particular interest to us is the case where M is a small knot exterior, for in this case, $X(M)$ has dimension 1 [7].

An algebraic component X_0 of $X(\Gamma)$ is called *nontrivial* if it contains the character of an irreducible representation and *trivial* otherwise. Since the character of a reducible representation in $R(\Gamma)$ is also the character of a diagonal representation, the number of trivial algebraic components of $X(\Gamma)$ can be easily determined from the first homology of Γ. Thus we shall concentrate on the non-trivial components.

As an example, suppose that M is a knot exterior which admits the structure of a Seifert fibred space with base orbifold $\mathcal{B}$. Using the fact that an irreducible representation $\pi_1(M) \to SL(2,\mathbb{C})$ factors through $\pi_1^{orb}(\mathcal{B})$ (a product of cyclic groups), or a degree 2 extension of this group, it is not difficult to determine the number of nontrivial algebraic components of $X(M)$. However when M is a hyperbolic knot exterior, such a determination appears to be more difficult. The fundamental groups of such manifolds admit discrete faithful representations to $SL(2,\mathbb{C})$, which are irreducible, and it was proven in Chapter 1 of [10] that any component of $X(M)$ which contains the character of such a representation is 1-dimensional. When M is the exterior of a hyperbolic twist knot in S^3, then one can deduce from [6] that $X(M)$ has exactly one nontrivial component. The same is true for the exterior of the $(-2,3,7)$-pretzel knot [1], or more generally the $(-2,3,n)$-pretzel (n odd) when it is hyperbolic and $n \not\equiv 0 \pmod 3$ [19]. Examples of small hyperbolic knot exteriors with character varieties containing at least two nontrivial algebraic components were obtained in [6], [21], [16], [22]; they are the exteriors of certain 2-bridge knots in S^3. See also [19]. In this paper we provide some general methods for producing hyperbolic knot exteriors with a large number of nontrivial algebraic components in their character varieties.

Given a slope α on ∂M, we use $M(\alpha)$ to denote the manifold obtained by Dehn filling M with the slope α. Note that there are natural inclusions $R(M(\alpha)) \subset R(M)$ and $X(M(\alpha)) \subset X(M)$.

If a knot exterior M contains an orientable, properly embedded surface F with exactly one boundary component which is essential on ∂M, we call it a *knot exterior with Seifert surface*. (M is with Seifert surface if and only if the composition $H_1(\partial M) \to H_1(M) \to H_1(M)/\mathrm{Torsion}(H_1(M))$ is onto.) Given a knot exterior M with Seifert surface F, one can construct the n-sheeted free cyclic cover M_n of M, dual to the surface F, for each integer $n > 1$. The boundary of the Seifert surface in M_n is called the *longitudinal class* of M_n and will be denoted by λ_n. In §3 we will show

Theorem A *Let M be a small knot exterior with Seifert surface and consider a sequence of positive integers $1 \leq a_1 < a_2 < \ldots < a_k < \ldots$ where each a_k divides a_{k+1}. Suppose that for each $k \geq 1$*
(a) M_{a_k} is small knot exterior;
(b) the number of irreducible characters in $X(M_{a_k}(\lambda_{a_k}))$ is finite but increases to ∞ with k;
(c) no irreducible representation $\rho \in R(M_{a_k}(\lambda_{a_k})) \subset R(M_{a_k})$ kills $\pi_1(\partial M_{a_k})$, i.e. $\rho(\pi_1(\partial M_{a_k}))$ is not contained in $\{\pm I\}$, where I is the unit matrix in $SL_2(\mathbb{C})$.
Then the number of nontrivial curve components in $X(M_{a_k})$ increases to ∞ with k.

The proof of the theorem is based on a study of the relationship between the character varieties $X(M_{a_k})$ induced by the covering maps $M_{a_j} \to M_{a_k}$ where $j \geq k$. More precisely, these covers induce maps $X(M) \to X(M_{a_1}) \to \ldots \to X(M_{a_k}) \to X(M_{a_{k+1}}) \to \ldots$ which we shall refer to, henceforth, as *restrictions*. We find that under the hypothesized conditions, certain mutually distinct, nontrivial components of $X(M_{a_k})$ restrict to mutually distinct, non-trivial components of $X(M_{a_j})$ for each $j \geq k$. Moreover, for $j \gg k$, $X(M_{a_j})$ contains nontrivial components which do not arise from restriction.

According to a theorem of D. Cooper and D. Long [8], any hyperbolic manifold M which satisfies the hypotheses of the theorem necessarily fibres over the circle. We can find many fibred knot exteriors satisfying the conditions of Theorem A. In particular, in §4 we determine $X(M(\lambda))$ when M is a hyperbolic punctured torus bundle over the circle S^1 (Proposition 4.5) and consequently obtain,

Theorem B *Let M be a hyperbolic punctured torus bundle over S^1. Then for any sequence of positive integers $1 \leq a_1 < a_2 < \ldots < a_k < \ldots$ where each a_k divides a_{k+1}, the number of curve components in $X(M_{a_k})$ approaches ∞ with k.*

With a view to refining our analysis, let Γ be a finitely generated group and $\gamma \in \Gamma$. Consider the regular function

$$\tau_\gamma : X(\Gamma) \to \mathbb{C}, \quad \tau_\gamma(\chi_\rho) = \chi_\rho(\gamma) = trace(\rho(\gamma)),$$

called the *trace function* on $X(\Gamma)$ defined by γ. Elementary trace identities imply that if γ' is either the inverse of γ or conjugate to γ in Γ, then $\tau_{\gamma'} = \tau_\gamma$. We say a subset Y of $X(\Gamma)$

is τ_γ-*non-constant* if $\tau_\gamma|Y$ is non-constant. In order to simplify the presentation, we shall frequently use τ_γ to denote $\tau_\gamma|Y$.

Since we have assumed our knot exteriors to be boundary-irreducible, there is an injective homomorphism $H_1(\partial M) \cong \pi_1(\partial M) \to \pi_1(M)$, well-defined up to conjugation. Hence each element $\delta \in \pi_1(\partial M) \cong H_1(\partial M)$ unambiguously determines a trace function τ_δ on $X(M)$. It is an observation made in [3], based on the work of [12] and [10], that for a knot exterior M, each (irreducible) curve $X_0 \subset X(M)$ belongs to one of the following three mutually exclusive types:

(i) The curve X_0 is τ_δ-non-constant for every nontrivial element $\delta \in \pi_1(\partial M)$;

(ii) The function τ_δ is a constant function on X_0 for every $\delta \in \pi_1(\partial M)$ (this case cannot arise if M is small, cf. Lemma 2.1);

(iii) There is exactly one primitive element δ_0, up to taking inverses, in $\pi_1(\partial M)$, such that τ_{δ_0} is a constant function on X_0.

We will give a proof of this statement below (Proposition 2.2) in our current $SL(2, \mathbb{C})$ setting.

If $X_0 \subset X(M)$ is a curve of type (iii), then the slope δ_0 is a boundary slope (Lemma 2.1) and we call it the boundary-slope associated to X_0. In §6, we concentrate on the hyperbolic punctured torus bundles which arise as cyclic covers of the figure-eight knot exterior. We give a complete list of nontrivial curve components, classified according to the three types (i)-(iii) above, in the character variety of the double cover and of the triple cover of the figure-eight knot exterior. For instance, the character variety of the 3-fold cover has precisely four nontrivial curve components of type (i) and six nontrivial curve components of type (iii) whose associated boundary slopes are the meridian slope of the bundle. The calculations there also produce the following unexpected phenomenon (see §6).

Theorem C *For every given integer m, there exists a small hyperbolic knot exterior M such that $X(M)$ contains two type (iii) curve components whose associated boundary slopes have distance (i.e. their minimal geometric intersection number in ∂M) larger than m.*

If $X_0 \subset X(M)$ is a curve of type (i), then one can use it to establish a norm on the 2-dimensional real vector space $H_1(\partial M; \mathbb{R})$ [10], [3]. So we shall also call a type (i) curve in $X(M)$ a *norm curve*. We have mentioned that if M is a hyperbolic knot exterior, then there is at least one nontrivial curve component X_0 in $X(M)$, one which contains the character of a discrete faithful representation of $\pi_1(M)$. We further note here that X_0 is in fact a norm curve [10] and the norm associated to this curve plays a crucial role in proving the cyclic surgery theorem of [10] and the finite surgery theorem of [2,3]. Our final goal in this paper is to show that there is no upper bound either on the number of norm curve components in the character variety of a hyperbolic knot exterior (§7).

Theorem D *For every given integer m, there exists a hyperbolic knot exterior M such that $X(M)$ contains more than m norm curve components, each of which contains the character of a discrete faithful representation of $\pi_1(M)$.*

Our studies suggest the following open questions for further investigation.

Questions:

(1) Given a hyperbolic knot exterior M, is it true that for any integer m, there always exists an n-fold cyclic cover of M whose character variety has more than m nontrivial algebraic components?

(2) Given a hyperbolic knot exterior M, is it true that for any integer m, there always exists an n-fold cyclic cover of M which has more than m distinct boundary slopes?

(3) For any integer m, is there a hyperbolic knot exterior in S^3 whose character variety has more than m type (i) curve components?

(4) For any integer m, is there a hyperbolic knot exterior in S^3 whose character variety has two type (iii) curve components whose associated boundary slopes have distance larger than m?

Our basic references for standard terminology and facts are [15], [17], [23] for 3-manifold topology and knot theory, [25] for algebraic geometry, and [12] for $SL(2,\mathbb{C})$-character varieties of 3-manifolds.

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2 Some properties of character varieties of knot exteriors

In this section we prepare some general results concerning the character varieties of knot exteriors which will be needed in later sections.

By an *essential surface* in a compact orientable 3-manifold W, we mean a properly embedded orientable incompressible surface, no component of which is either boundary parallel or bounds a 3-cell in W.

Let M be a knot exterior. A *slope* on ∂M is the ∂M-isotopy class of an unoriented simple closed essential curve in ∂M. For notational simplicity, we shall use the same symbol to denote a slope, the corresponding primitive element in $H_1(\partial M) = \pi_1(\partial M)$ (well-defined up to sign); the context will make it clear which is meant.

If δ is a slope, then $M(\delta)$ will denote the manifold obtained by Dehn filling M along ∂M with the slope δ (i.e. a solid torus V is attached to M along their boundaries so that the slope δ bounds a meridian disk of V). A slope of a knot exterior is called a *boundary slope* if there is a connected essential surface F in M such that ∂F is a non-empty set of curves in ∂M of the given slope r.

For an irreducible complex affine algebraic curve X_0, we use $\tilde{X}_0$ to denote its smooth projective completion. Note that $\tilde{X}_0$ is birationally equivalent to X_0. Since such a birational equivalence induces an isomorphism between the function fields $\mathbb{C}(X_0)$ and $\mathbb{C}(\tilde{X}_0)$, any rational function f on X_0 determines a rational function on $\tilde{X}_0$, which will also be denoted f. A point of $\tilde{X}_0$ is called an *ideal* point if it is a pole of some $f \in \mathbb{C}[X_0] \subset \mathbb{C}(X_0) \cong \mathbb{C}(\tilde{X}_0)$.

Consider the case where X_0 is a curve in the $SL(2,\mathbb{C})$-character variety $X(M)$ of a knot exterior M. One of the fundamental connections between the topology and the character variety of M, found by Culler and Shalen [12], can be briefly described as follows (see [12] for details):

Start with an ideal point x_0 of $\tilde{X}_0$. It determines a discrete valuation v on the function field $\mathbb{K} = \mathbb{C}(X_0) \cong \mathbb{C}(\tilde{X}_0)$ whose valuation ring consists of those elements of $\mathbb{K}$ which do not have pole at the point x_0. Choose an algebraic component $R_0 \subset R(M)$ of $t^{-1}(X_0)$ such that $t|R_0$ is not constant and extend v to a discrete valuation on the function field $\mathbb{F} = \mathbb{C}(R_0)$. According to Bass-Serre [24], one can construct a simplicial tree on which $SL(2,\mathbb{F})$ acts. There is a *tautological representation* $P : \pi_1(M) \to SL(2,\mathbb{F})$, which then induces an action of $\pi_1(M)$ on the tree. It is useful to observe that the tautological representation satisfies the identity

$$trace(P(\gamma)(\rho)) = \tau_\gamma(\chi_\rho) \quad \gamma \in \pi_1(M), \rho \in R_0.$$

An element $\gamma \in \pi_1(M)$ fixes a vertex of the tree if and only if x_0 is not a pole of τ_γ, and so using the fact that x_0 is an ideal point of X_0, it can be shown that the action of $\pi_1(M)$ is nontrivial, i.e. no vertex of the tree is fixed by the entire group $\pi_1(M)$. Hence the action yields a nontrivial splitting of $\pi_1(M)$ as the fundamental group of a graph of groups. This splitting of the group in turn yields a splitting of the manifold M along essential surfaces. We shall say that such essential surfaces in M are *associated* to the ideal point x_0. If C is a connected subcomplex of ∂M and x_0 is not a pole of τ_δ for any $\delta \in \pi_1(C)$, then there exists an essential surface in M associated to x_0 that is disjoint from C; and if x_0 is a pole of τ_δ for some slope δ in ∂M, then any essential surface associated to x_0 must intersect δ. Therefore the following lemma holds.

Lemma 2.1 *Let M be a knot exterior, $X_0 \subset X(M)$ a curve, and $x_0 \in \tilde{X}_0$ an ideal point.*
(1) *If x_0 is not a pole of τ_α for some slope α, but is a pole of τ_β for another slope β, then α is a boundary slope.*

(2) *If x_0 is not a pole of τ_α and τ_β for two different slopes α and β, then there exists a closed essential surface in M associated to x_0.* $\diamondsuit$

Proposition 2.2 *Let M be a knot exterior and $X_0 \subset X(M)$ a curve. Then X_0 belongs to one of the three mutually exclusive types (i)-(iii) described in §1.*

Proof. Suppose that X_0 is neither of type (i) nor (ii). Then there are two nontrivial elements $\alpha_0, \alpha_1 \in \pi_1(\partial M)$ such that τ_{α_0} is a constant function on X_0 but τ_{α_1} is not.

<u>Claim</u> For any element $\gamma \in \pi_1(M)$ and integer $n \neq 0$, τ_γ is constant on X_0 if and only if τ_{γ^n} is.

<u>Proof of Claim</u> Since $\tau_{\gamma^n} = \tau_{\gamma^{-n}}$, we may suppose that $n > 0$. The trace identity

$$trace(A)trace(B) = trace(AB) + trace(AB^{-1}) \qquad A, B \in SL(2, \mathbb{C}),$$

implies that for any $\chi_\rho \in X_0$, $\tau_{\gamma^2}(\chi_\rho) = trace(\rho(\gamma^2)) = trace((\rho(\gamma))^2) = [trace(\rho(\gamma))]^2 - 2$. Thus τ_γ is constant on X_0 if and only τ_{γ^2} is a constant function on X_0. A simple induction argument, based on the trace identity, can now be constructed to see that τ_γ is constant on X_0 if and only if τ_{γ^n} is constant on X_0 for any integer $n > 0$, and hence for each $n \neq 0$. This proves the Claim.

By the claim, we may assume that the elements α_0, α_1 are primitive elements of $\pi_1(\partial M)$. It follows from Lemma 2.1 (1) that for any slope δ on ∂M, if τ_δ is constant on X_0, then δ is a boundary slope. By [14], there are only finitely many boundary slopes on ∂M and therefore there must exist a slope β_0 on ∂M such that $\{\alpha_0, \beta_0\}$ is a basis for $\pi_1(\partial M)$ but β_0 is not a boundary slope. It follows that X_0 is τ_{β_0}-non-constant.

Let R_0 be an algebraic component of $t^{-1}(X_0)$ for which $t : R_0 \rightarrow X_0$ non-constant. Since X_0 is τ_{β_0}-non-constant, there is a field extension $\mathbb{E}$ of $\mathbb{F} = \mathbb{C}(R_0)$, of degree at most two, such that the tautological representation $P : \pi_1(M) \rightarrow SL(2, \mathbb{F})$ is conjugate over $GL(2, \mathbb{E})$ to a representation $P' : \pi_1(M) \rightarrow SL(2, \mathbb{E})$ and that $P'(\beta_0)$ is a diagonal matrix. Since α_0 and β_0 commute and since $P'(\beta_0) \neq \pm I$, $P'(\alpha_0)$ must also be a diagonal matrix, say

$$P'(\alpha_0) = \begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix} \quad \text{and} \quad P'(\beta_0) = \begin{pmatrix} b_0 & 0 \\ 0 & b_0^{-1} \end{pmatrix}.$$

From the identity $trace(P(\gamma)(\rho)) = \tau_\gamma(\chi_\rho)$, we have that for any $\delta = \alpha_0^m \beta_0^n \in \pi_1(\partial M)$, $\tau_\delta = a_0^m b_0^n + a_0^{-m} b_0^{-n}$.

Let x_0 be a pole of τ_{β_0} in $\tilde{X}_0$ (x_0 is necessarily an ideal point) and v be the discrete valuation on $\mathbb{K} = \mathbb{C}(\tilde{X}_0)$ defined by x_0. By [20, Lemma II.4.4], the discrete valuation v can be extended to a discrete valuation w on the field $\mathbb{E}$, i.e. $w|\mathbb{K} = dv$ for some integer

$d > 0$. It is easy to check that for any $a \in \mathbb{E} \setminus \{0\}$, $w(a + a^{-1}) < 0$ if and only if $w(a) \neq 0$, and so by construction $w(a_0) = 0, w(b_0) \neq 0$. Hence if $\delta = a_0^m b_0^n \in \pi_1(\partial M)$, then $0 \neq w(\delta) = w(a_0^m b_0^n) = nw(b_0)$ if and only if $n \neq 0$; that is, $v(\tau_\delta) = \frac{1}{d} w(\tau_\delta) < 0$ if and only if $n \neq 0$. Put another way, x_0 is a pole of τ_d if and only if $n \neq 0$. It follows that X_0 is a τ_δ-non-constant curve if and only if $n \neq 0$. Thus X_0 is of type (iii) and the lemma has been proven. $\diamondsuit$

The next proposition is a special case, sufficient for our needs, of a result in [26], [12] concerning the dimensions of components of $X(M)$ for a general compact 3-manifold with boundary.

Proposition 2.3 ([26], [12]) *Let M be a knot exterior. If $\chi_\rho \in X(M)$ is the character of an irreducible representation $\rho \in R(M)$ such that $\rho(\pi_1(\partial M))$ is not contained in $\{\pm I\}$, then any component of $X(M)$ containing χ_ρ has dimension at least* 1. $\diamondsuit$

Proposition 2.4 *Let M be a small knot exterior.*
(1) [10] *Each component of $X(M)$ is at most* 1*-dimensional.*
(2) *Every* 1*-dimensional component X_0 of $X(M)$ is either a type (i) or type (iii) curve.*

Proof. The argument for part (1) can be found in the proof of [7, Proposition 2.4]. To prove part (2), suppose otherwise that X_0 has type (ii) (Proposition 2.2). Then for any ideal point x_0 of $\tilde{X}_0$ and $\alpha \in \pi_1(\partial M)$, x_0 is not a pole of τ_α. It follows from Lemma 2.1 (2) that there is a closed, essential surface in M. But this contradicts our assumption that M is a small knot exterior. $\diamondsuit$

Proposition 2.5 *Suppose that M is a small knot exterior and that λ is a primitive element of $H_1(\partial M)$ such that $X(M(\lambda))$ contains no nontrivial component of positive dimension. Then τ_λ cannot be identically equal to* 2 *on any nontrivial positive dimensional component of $X(M)$.*

Proof. Suppose otherwise that there is a nontrivial positive dimensional component X_0 of $X(M)$ on which τ_λ is constantly equal to 2. According to Proposition 2.4 (1), X_0 is a curve. Recall $t : R(M) \to X(M)$ the regular, surjective map which sends a representation to its character. Let R_0 be an algebraic component of $t^{-1}(X_0)$ on which t is non-constant. Then every representation in R_0 maps λ to a matrix of trace 2.

Claim $\rho(\lambda) = I$ for every $\rho \in R_0$.

Proof of the Claim We only need to show the claim for irreducible representations since they form a dense subset of R_0. Suppose otherwise that there is an irreducible representation

$\rho_0 \in R_0$ such that $\rho_0(\lambda)$ is not the identity matrix. It follows from [12, Proposition 1.5.4] that there is a Zariski open neighborhood U of ρ_0 in R_0 such that $\rho(\lambda) \neq I$ for each $\rho \in U$. Thus $\rho(\lambda)$ must be a trace 2 parabolic element of $SL(2,\mathbb{C})$ for each $\rho \in U$. If μ is any element of $\pi_1(\partial M)$, then as it commutes with λ, $\rho(\mu)$ is either parabolic or $\pm I$ for each $\rho \in U$. Hence τ_μ is also constantly equal to either 2 or -2 on $t(U)$, and therefore on X_0. This is impossble as it contradicts Proposition 2.4 (2). Thus the claim holds.

According to this claim, $R_0 \subset R(M(\lambda))$ and therefore X_0 is contained in $X(M(\lambda))$, contrary to our hypothesis that $X(M(\lambda))$ does not contain any nontrivial positive dimensional components. The proposition is proved. $\diamondsuit$

Proposition 2.6 *Suppose that M is a knot exterior such that some primitive element λ of $H_1(\partial M)$ is zero in $H_1(M)$. Suppose also that $X_0 \subset X(M)$ is an algebraic component on which τ_λ is non-constant. Then X_0 is a nontrivial component of $X(M)$ of positive dimension.*

Proof. Since $\lambda = 0$ in $H_1(M)$, we have $H_1(M(\lambda)) = H_1(M)$. Any character of a reducible representation of $\pi_1(M)$ is also the character of a diagonal representation of $\pi_1(M)$, and thus is the character of a representation which factors through $H_1(M) = H_1(M(\lambda)))$. Therefore any trivial component of $X(M)$ is also contained in $X(M(\lambda))$. Clearly τ_λ is constantly equal to 2 on $X(M(\lambda))$, and so the proposition follows. $\diamondsuit$

3 Character varieties and covering spaces

In this section we study relations between the character varieties of covering spaces. This will lead us to a proof of Theorem A.

Let M be a knot exterior and let $p : M_n \to M$ an n-fold regular (free) covering. The map p induces an injective homomorphism $p_* : \pi_1(M_n) \to \pi_1(M)$ whose image is an index n normal subgroup of $\pi_1(M)$. The homomorphism p_* in turn induces regular maps

$$p^* : R(M) \to R(M_n), \quad p^*(\rho) = \rho \circ p_*,$$

where "$\circ$" denotes composition, and

$$\hat{p} : X(M) \to X(M_n), \quad \hat{p}(\chi_\rho) = \chi_{p^*(\rho)}.$$

We have the following commutative diagram of regular maps, the two vertical ones being surjective:

$$\begin{array}{ccc} R(M) & \xrightarrow{p^*} & R(M_n) \\ t \downarrow & & \downarrow t \\ X(M) & \xrightarrow{\hat{p}} & X(M_n). \end{array}$$

For any subset R_0 of $R(M)$, we call the Zariski closure of $p^*(R_0)$ in $R(M_n)$ the *restriction* subvariety of R_0 in $R(M_n)$. Similarly for $X_0 \subset X(M)$, the *restriction* subvariety of X_0 in $X(M_n)$ is the Zariski closure of $\hat{p}(X_0)$ in $X(M_n)$.

Proposition 3.1 *If $X_0 \subset X(M)$ is an algebraic curve, then $\hat{p}(X_0)$ is a (closed) algebraic curve in $X(M_n)$.*

Proof. Let $\overline{\hat{p}(X_0)}$ denote the restriction subvariety of X_0 in $X(M_n)$. Since the restriction of $\hat{p}| : X_0 \to \overline{\hat{p}(X_0)}$ is dominating, the dimension of $\overline{\hat{p}(X_0)}$ is bounded above by 1. On the other hand, if x is an ideal point of X_0, there is some $\delta \in \pi_1(M)$ such that x is a pole of τ_δ. But then x is also a pole of δ^n (cf. the proof of the claim in the proof of Proposition 2.2). Since $\delta^n \in \pi_1(M_n)$, it follows that τ_{δ^n} is non-constant on $\overline{\hat{p}(X_0)}$, which therefore has dimension 1. It also follows that the induced surjection between the smooth projective models of X_0 and $\overline{\hat{p}(X_0)}$ sends ideal points to ideal points. Thus $\overline{\hat{p}(X_0)} = \hat{p}(X_0)$. $\diamondsuit$

Proposition 3.2 *Suppose M is a small knot exterior and that some primitive element λ of $H_1(\partial M)$ is zero in $H_1(M)$. Let $p : M_n \to M$ be a free n-fold cyclic covering such that M_n is also a small knot exterior, and suppose that λ_n is the primitive element of $\pi_1(\partial M_n)$ such that $p_*(\lambda_n) = \lambda$. If $X_0 \subset X(M)$ is a τ_λ-non-constant component, then the restriction subvariety Y_0 of X_0 in $X(M_n)$ is a nontrivial, τ_{λ_n}-non-constant curve component of $X(M_n)$.*

Proof. By Propositions 2.4 and 2.6, X_0 is a nontrivial curve component of $X(M)$. It follows as in Lemma 4.1 of [3] that there is a 4-dimensional algebraic component R_0 of $R(M)$ such that $t(R_0) = X_0$. If we denote by S_0 the restriction subvariety of R_0 on $R(M_n)$, then S_0 is necessarily irreducible and $p^*(R_0)$ contains a Zariski open subset of S_0. Similarly if we let Y_0 be the restriction subvariety of X_0 in $X(M_n)$, then Y_0 is irreducible and $\hat{p}(X_0) = t(p^*(R_0))$ is a dense subset of Y_0. Since the covering is cyclic and $\lambda = 0$ in $H_1(M)$, the simple closed curve λ lifts to n parallel, simple closed curves in ∂M_n, each representing $\lambda_n \in \pi_1(\partial M_n)$. Observe that for $\chi_\rho \in X_0$,

$$\tau_\lambda(\chi_\rho) = trace(\rho(\lambda)) = trace(\rho(p_*(\lambda_n))) = \tau_{\lambda_n}(t(p^*(\rho))),$$

and therefore Y_0 is τ_{λ_n}-non-constant. Hence by Proposition 2.6, the component of $X(M_n)$ containing Y_0 is nontrivial. Now applying Proposition 2.4, we see that Y_0 is itself a curve component of $X(M_n)$. $\diamondsuit$

Proposition 3.3 *Let Γ be a finitely generated group and Γ_0 a normal subgroup. Suppose that $\chi_{\rho_1}, \chi_{\rho_2} \in X(\Gamma)$ both restrict to the same irreducible charcter of Γ_0. Then there is a homomorphism $\epsilon : \Gamma \to \{\pm I\}$, which vanishes on Γ_0, such that ρ_2 is conjugate to $\epsilon\rho_1$. Hence a given irreducible representation $\Gamma_0 \to SL(2, \mathbb{C})$ extends to no more than $\#H^1(\Gamma/\Gamma_0; \mathbb{Z}/2)$ representations in $R(M)$.*

Proof. By [12, Proposition 1.5.2] we may assume that ρ_1 and ρ_2 actually restrict to the same representation on Γ_0. The normality of Γ_0 in Γ implies that for each $\gamma \in \Gamma$ and $\sigma \in \Gamma_0$ we have

$$\rho_1(\gamma)\rho_1(\sigma)\rho_1(\gamma^{-1}) = \rho_1(\gamma\sigma\gamma^{-1}) = \rho_2(\gamma\sigma\gamma^{-1}) = \rho_2(\gamma)\rho_2(\sigma)\rho_2(\gamma^{-1}) = \rho_2(\gamma)\rho_1(\sigma)\rho_2(\gamma^{-1}).$$

Hence by the irreducibility of $\rho_1|\Gamma_0$, we see that $\rho_2(\gamma)^{-1}\rho_1(\gamma) = I$ or $-I$. Thus we have a map $\epsilon : \Gamma \to \{\pm I\}$, easily seen to be a homomorphism which vanishes on Γ_0, such that $\rho_1(\gamma) = \epsilon(\gamma)\rho_2(\gamma)$ for every $\gamma \in \Gamma$. This completes the proof. $\diamondsuit$

Remark 3.4 Recall from [3] that there is an action of $H^1(\Gamma; \mathbb{Z}/2) = Hom(\Gamma, \{\pm 1\})$ on both $R(\Gamma)$ and $X(\Gamma)$. Thus for $\epsilon \in H^1(\Gamma; \mathbb{Z}/2)$ there are algebraic isomorphisms $\epsilon^* : R(\Gamma) \to R(\Gamma), \hat{\epsilon} : X(\Gamma) \to X(\Gamma)$ given by

$$\epsilon^*(\rho)(g) = \epsilon(g)\rho(g), \quad \hat{\epsilon}(\chi_\rho)(g) = \chi_{\epsilon^*(\rho)}(g) = \epsilon(g)\chi_\rho(g).$$

The previous proposition may be interpreted as saying that the set of representations (character) of Γ which restrict to an *irreducible* representation (character) of Γ_0 is either empty or an orbit of the $H^1(\Gamma/\Gamma_0; \mathbb{Z}/2) \subset H^1(\Gamma; \mathbb{Z}/2)$ action.

Corollary 3.5 *M be a knot exterior and suppose that $p : M_n \to M$ is a free, n-fold cyclic covering. Let $\hat{p} : X(M) \to X(M_n)$ be the regular map induced by the covering map p. Then each irreducible character in $X(M_n)$ has at most two inverse images in $X(M)$ under the map $\hat{p}$.* $\diamondsuit$

Proposition 3.6 *Suppose that $p : M_n \to M$ is a free n-fold cyclic covering between knot exteriors. Suppose further that $\lambda \in H_1(\partial M)$ is zero in $H_1(M)$ and that λ_n is the primitive element of $\pi_1(\partial M_n)$ such that $p_*(\lambda_n) = \lambda$. If X_0 is a τ_λ-non-constant component of $X(M)$ and Y_0 its restriction in $X(M_n)$, then the degree of the function $\tau_\lambda : \tilde{X}_0 \to \mathbb{C}P^1$ is non-zero and is either equal to or the double of the degree of $\tau_{\lambda_n} : \tilde{Y}_0 \to \mathbb{C}P^1$.*

Proof. The regular dominating map $\hat{p} : X_0 \to Y_0$ induces the rational map $\tilde{p} : \tilde{X}_0 \longrightarrow \tilde{Y}_0$ such that the following diagram of maps commutes:

$$\begin{array}{ccc} \tilde{X}_0 & \xrightarrow{\tilde{p}} & \tilde{Y}_0 \\ \updownarrow & & \updownarrow \\ X_0 & \xrightarrow{\hat{p}} & Y_0, \end{array}$$

where the vertical arrows denote the birational isomorphisms. Now the commutative diagram of regular maps:

$$\begin{array}{ccccc} X_0 & & \xrightarrow{\hat{p}} & & Y_0 \\ & \tau_\lambda \searrow & & \swarrow \tau_{\lambda_n} & \\ & & \mathbb{C} & & \end{array}$$

induces a commutative diagram of rational maps:

$$\begin{array}{ccccc} \tilde{X}_0 & & \xrightarrow{\tilde{p}} & & \tilde{Y}_0 \\ & \tau_\lambda \searrow & & \swarrow \tau_{\lambda_n} & \\ & & \mathbb{C}P^1 & & . \end{array}$$

Therefore $\tau_\lambda = \tau_{\lambda_n} \circ \tilde{p}$ and so $degree(\tau_\lambda) = degree(\tilde{p}) degree(\tau_{\lambda_n})$. Since τ_λ is non-constant, $degree(\tilde{p}) > 0$. We need only show that $degree(\tilde{p} : \tilde{X}_0 \rightarrow \tilde{Y}_0) \in \{1, 2\}$ to complete the proof. But from the first diagram above, it is enough to show that there is a dense subset of Y_0 each point of which has no more than two inverse images in X_0 under the map $\hat{p}$. Since Y_0 is a nontrivial component (Proposition 3.2), the set of irreducible characters in Y_0 is dense [12], so an appeal to Corollary 3.5 completes the proof. $\diamondsuit$

Corollary 3.7 *Suppose that $p : M_n \rightarrow M$ is a free n-fold cyclic covering between knot exteriors and that some primitive element λ of $H_1(\partial M)$ is zero in $H_1(M)$. If X_0 and X_1 are two τ_λ-non-constant components in $X(M)$ such that the degree of τ_λ on $\tilde{X}_0$ is different from the degree of τ_λ on $\tilde{X}_1$, then the restriction subvarieties Y_0 and Y_1 of X_0 and X_1 on M_n are distinct curve components of $X(M_n)$.*

Proof. Let λ_n be the primitive element of $\pi_1(\partial M_n)$ such that $p_*(\lambda_n) = \lambda$. By the proof of Proposition 3.6, we have $degree(\tau_\lambda|_{\tilde{X}_0}) = degree(\tilde{p}|_{\tilde{X}_0}) degree(\tau_{\lambda_n}|_{\tilde{Y}_0})$ and $degree(\tau_\lambda|_{\tilde{X}_1}) = degree(\tilde{p}|_{\tilde{X}_1}) degree(\tau_{\lambda_n}|_{\tilde{Y}_1})$ Hence if $degree(\tilde{p}|_{\tilde{X}_0}) = degree(\tilde{p}|_{\tilde{X}_1})$, then $degree(\tau_{\lambda_n}|_{\tilde{Y}_0}) \neq degree(\tau_{\lambda_n}|_{\tilde{Y}_1})$ and so Y_0 and Y_1 are distinct curve components of $X(M_n)$. Without loss of generality then, we may assume, by Proposition 3.6, that $degree(\tilde{p}|_{\tilde{X}_0}) = 1$ and $degree(\tilde{p}|_{\tilde{X}_1})$ $= 2$. But now Y_0 and Y_1 cannot be the same component. For otherwise there would be three inverse images in $X_0 \cup X_1$ for a generic irreducible character of $Y_0 = Y_1$ under the map $\hat{p}$, one in X_0 and two in X_1, and this contradicts Corollary 3.5. $\diamondsuit$

Lemma 3.8 *Suppose that $p : M_n \rightarrow M$ is a free n-fold regular covering of knot exteriors and that X_0 is a curve component of $X(M)$. If X_0 has, respectively, type (i), (ii), or (iii), then its restriction Y_0 on M_n is a type (i), (ii) or (iii) curve component of $X(M_n)$ respectively. In the case where X_0 is a type (iii) curve whose associated boundary slope is δ, then the boundary slope associated to Y_0 is the slope of $p^{-1}(\delta) \subset \partial M_n$.*

Proof. Suppose that X_0 is a type (i) curve and let $\delta_n \in \pi_1(\partial M_n)$ be any nontrivial element. Then $p_*(\delta_n)$ is a nontrivial element of $\pi_1(\partial M)$ and by a similar argument to that used in proving Proposition 3.6 we have

$$degree(\tau_{p_*(\delta_n)}|_{\tilde{X}_0}) = degree(\tilde{p}|_{\tilde{X}_0})degree(\tau_{\delta_n}|_{\tilde{Y}_0}).$$

Now since X_0 is a type (i) curve, we have $degree(\tau_{p_*(\delta_n)}|_{\tilde{X}_0}) > 0$, and therefore $degree(\tau_{\delta_n}|_{\tilde{Y}_0}) > 0$, i.e. τ_{δ_n} is non-constant on Y_0. Thus Y_0 is a type (i) curve.

The cases when X_0 is a type (ii) or (iii) curve are proven similarly. $\diamondsuit$

Proof of Theorem A

We shall proceed by contradiction. Suppose that the theorem does not hold. Then after possibly passing to a subsequence of $\{a_k\}$, we may assume that there is some $N > 0$ such that for each M_{a_k}, the number of nontrivial curve components in $X(M_{a_k})$ is bounded above by N.

Fix $k \geq 1$ and suppose that there are exactly j_k mutually distinct nontrivial, $\tau_{\lambda_{a_k}}$-non-constant curve components $X_1, ..., X_{j_k}$ of $X(M_{a_k})$. Among these curves, assume that there are i_k of them, say $X_1, ..., X_{i_k}$, and no more, which satisfy the condition that when restricted to any cyclic cover M_{a_l}, $l \geq k$, they always yield i_k mutually distinct, nontrivial curve components $Y_1, ..., Y_{i_k}$ of $X(M_{a_l})$. Note that $Y_1, ..., Y_{i_k}$ are $\tau_{\lambda_{a_l}}$-non-constant by proposition 3.2. By our choice of N, there is some k_1 for which $i_k \leq i_{k_1}$ for all k. Set $i = i_{k_1}, j = j_{k_1}$, and $n_1 = a_{k_1}$. Let $X_1, \ldots, X_i, X_{i+1}, \ldots, X_j$ be the nontrivial, $\tau_{\lambda_{n_1}}$-non-constant curve components components of $X(M_{n_1})$, ordered in the fashion described above.

We claim that $i > 0$. By condition (b) of the hypotheses, there is a $k \geq 1$ for which $X(M_{a_k})(\lambda_{a_k})$ contains an irreducible character χ_1. Since $\tau_{\lambda_{a_k}}(\chi_1) = 2$, Proposition 2.5 implies that $\tau_{\lambda_{a_k}}$ is non-constant on such a curve. Now applying Proposition 3.2, we see that the restriction of such a curve to M_{a_l}, $l \geq k$, is a nontrivial, $\tau_{\lambda_{a_l}}$-non-constant curve component of $X(M_{a_l})$. Thus $i > 0$.

By Corollaries 3.5 and 3.7 and the defining choice of i, each of the integers

$$degree(\tau_{\lambda_{n_1}}|_{\tilde{X}_{i+1}}), degree(\tau_{\lambda_{n_1}}|_{\tilde{X}_{i+2}}), \ldots, degree(\tau_{\lambda_{n_1}}|_{\tilde{X}_j})$$

is equal to one of

$$degree(\tau_{\lambda_{n_1}}|_{\tilde{X}_1}), degree(\tau_{\lambda_{n_1}}|_{\tilde{X}_2}), \ldots, degree(\tau_{\lambda_{n_1}}|_{\tilde{X}_i}).$$

Set $q = \max\{degree(\tau_{\lambda_{n_1}}|_{\tilde{X}_l}) \mid 1 \leq l \leq i\}$ and let $n_2 = a_{k_2}$ where $k_2 > k_1$ is chosen so that the number of irreducible characters in $X(M_{n_2}(\lambda_{n_2}))$ is larger than Nq. If $Y_1, ..., Y_i$ are the restrictions to M_{n_2} of $X_1, ..., X_i$, then our assumptions imply that $Y_1, ..., Y_i$ are mutually distinct nontrivial, $\tau_{\lambda_{n_2}}$-non-constant curve components of $X(M_{n_2})$. Let $Y_1, ..., Y_i, Y_{i+1}, ..., Y_{j'}$

be the complete collection of such curves in $X(M_{n_2})$. Again by Corollaries 3.5 and 3.7 and the defining choice of i, each integer

$$degree(\tau_{\lambda_{n_2}}|_{\tilde{Y}_{i+1}}), \ldots, degree(\tau_{\lambda_{n_2}}|_{\tilde{Y}_{j'}})$$

is equal to one of

$$degree(\tau_{\lambda_{n_2}}|_{\tilde{Y}_1}), \ldots, degree(\tau_{\lambda_{n_2}}|_{\tilde{Y}_i})$$

and by Proposition 3.6 we have

$$degree(\tau_{\lambda_{n_2}}|_{\tilde{Y}_1}) \leq degree(\tau_{\lambda_{n_1}}|_{\tilde{X}_1}), \ldots, degree(\tau_{\lambda_{n_2}}|_{\tilde{Y}_i}) \leq degree(\tau_{\lambda_{n_1}}|_{\tilde{X}_i}).$$

Thus $degree(\tau_{\lambda_{n_2}}|_{\tilde{Y}_l}) \leq q$ for each $l \in \{1, 2, \ldots, j'\}$. But then as $\tau_{\lambda_{n_2}}$ takes the value 2 at each of the irreducible characters in $X(M_{n_2}(\lambda_{n_2})) \subset X(M_{n_2})$, it follows that no Y_l contains more than q of these characters. By construction, the number of irreducible characters in $X(M_{n_2})$ is larger than $Nq \geq j'q$ and so at least one such character, χ say, is not contained in $Y_1 \cup \ldots \cup Y_{j'}$. But then by Propositions 2.3 and 2.4, χ is contained in a nontrivial, $\tau_{\lambda_{n_2}}$-non-constant curve component $Y_{j'+1}$ of $X(M_{n_2})$, contrary to the definition of j'. Thus the theorem must hold. $\diamondsuit$

4 Character varieties of torus bundles over S^1

We call an $SL(2, \mathbb{C})$-character *binary dihedral* if it is the character of a representation whose image is a nonabelian binary dihedral group. In this section we show that each irreducible $SL(2, \mathbb{C})$-character of torus bundle over S^1 with hyperbolic monodromy is binary dihedral and obtain an exact count of their number. Together with results from [13] and [11], we show that any hyperbolic punctured torus bundle satisfies all the conditions of Theorem A. Hence we obtain Theorem B.

Consider a torus $T = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$ and fix base points $(1, 1) \in T$ and $(0, 0) \in \mathbb{R}^2$. The action of $GL_2(\mathbb{Z})$ on $\mathbb{R}^2$ descends to one on T in such a way that under the natural identification $H_1(T) = \mathbb{Z}^2 \subset \mathbb{R}^2$, the diffeotopy group of T is isomorphic, in the obvious way, to $GL_2(\mathbb{Z}) = Aut(H_1(T))$. An element of this group is called *hyperbolic* if its trace is larger than 2 in absolute value.

Fix an orientation preserving diffeomorphism $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ of T and let W be the torus bundle over the circle with monodromy h. Since h has a fixed point, the trace of this point is a closed loop in W. Throughout this section we shall use μ to denote either this loop, or its class in $\pi_1(W)$ or $H_1(W)$, and in all cases refer to it as a *meridian* of W.

Evidently μ is sent to a generator of $H_1(S^1)$ under the projection-induced homomorphism $H_1(W) \to H_1(S^1)$.

Consider the endomorphism

$$h_* - 1_{H_1(T)} : H_1(T) \to H_1(T).$$

Lemma 4.1 *Let W be a torus bundle over S^1 with monodromy h.*
(1) $H_1(W) \cong \mathbb{Z} \oplus coker(h_* - 1_{H_1(T)})$ where the $\mathbb{Z}$-factor is generated by μ.
(2) $\# coker(h_* - 1_{H_1(T)}) = |2 - trace(h)|$ if $trace(h) \neq 2$.

Proof. (1) Let $N(T) \subset W$ be a collar neighbourhood of T and set $W_0 = W \setminus \mathrm{int}(N(T))$. Evidently $W_0 \cong T \times I$. The isomorphisms $H_j(W,T) = H_j(W, N(T))$ (homotopy) $\cong H_j(W_0, \partial W_0)$ (excision) $\cong H_{j-1}(T)$ (Thom isomorphism) can be used to convert the exact sequence

$$H_2(W,T) \to H_1(T) \to H_1(W) \to H_1(W,T) \to H_0(T) \to H_0(W)$$

to

$$H_1(T) \xrightarrow{h_* - 1} H_1(T) \to H_1(W) \to \mathbb{Z} \to 0.$$

Part (1) follows.

(2) Part (2) follows from the identity $|\det(h_* - 1_{H_1(T)})| = |2 - \mathrm{trace}(h)|$. $\diamondsuit$

Proposition 4.2 *If W is a torus bundle over S^1 with monodromy h then*

$$H_1(W) \cong \begin{cases} \mathbb{Z} \oplus Torsion & \text{if } trace(h) \neq 2 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus Torsion & \text{if } trace(h) = 2 \text{ and } h \neq I \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \text{if } h = I \end{cases}$$

where the first $\mathbb{Z}$-factor is generated by μ and the rest is $coker(h_* - 1_{H_1(T)})$.

Proof. The proposition is a consequence of the previous lemma and the following observation: if $\mathrm{trace}(h) = 2$ and $h \neq I$, then h is conjugate in $SL_2(\mathbb{Z})$ to a matrix of the form $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for some $n \in \mathbb{Z} \setminus \{0\}$. $\diamondsuit$

Our next goal is to determine the $SL(2,\mathbb{C})$-character variety of W. To that end let $\rho \in R(W)$ be irreducible and observe that $\rho|\pi_1(T)$ is reducible. If $\rho|\pi_1(T)$ is central, then $\rho(\pi_1(T)) \subset \{\pm I\}$ and so as $\rho(\pi_1(W))$ is generated by $\rho(\mu)$ and $\rho(\pi_1(T))$, ρ is abelian, and therefore reducible, contrary to our hypotheses. Thus $\rho|\pi_1(T)$ is non-central and so there are exactly one or two $\rho|\pi_1(T)$-invariant lines in $\mathbb{C}^2$. Since $\pi_1(T)$ is normal in $\pi_1(W)$, the union of these lines is actually ρ-invariant. Since ρ is irreducible there must be two lines, and so a standard argument now implies that ρ is conjugate to a representation with image in

$$N = \{\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \begin{pmatrix} 0 & w \\ -w^{-1} & 0 \end{pmatrix} \mid z, w \in \mathbb{C}^*\}.$$

We can be more precise. Assume that ρ has image in N. Since there are two $\rho|\pi_1(T)$-invariant lines, $\rho|\pi_1(T)$ is diagonalisable, and so $\rho(\pi_1(T))$ consists of diagonal matrices. It follows that $\rho(\mu)$ must be a non-diagonal element of N. Thus $\rho(\mu)$ has order 4. Note that ρ can be conjugated by a diagonal matrix so that

$$\rho(\mu) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proposition 4.3 *Let W be a torus bundle over S^1 with monodromy h. An irreducible representation $\pi_1(W) \to SL(2, \mathbb{C})$ is conjugate to a representation with image in N. Furthermore we can assume that $\rho(\pi_1(T)) \subset D$ and $\rho(\mu) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.* $\diamondsuit$

We can be even more specific. Let W_2 be the torus bundle with monodromy h^2. Its fundamental group is presented by

$$\pi_1(W_2) = < \mu_2, \pi_1(T) \mid \mu_2 \gamma \mu_2^{-1} = h^2(\gamma) \text{ for all } \gamma \in \pi_1(T) >,$$

where μ_2 is the meridian of the T-bundle W_2. The natural covering projection $p_2 : W_2 \to W$ sends μ_2 to $\mu^2 \in \pi_1(W)$, and is the identity on $\pi_1(T)$.

It is a consequence of Proposition 4.3 that any irreducible character of $\pi_1(W)$ is the character of a representation $\rho : \pi_1(W) \to N$ such that $\rho(\mu) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\rho(\pi_1(W_2)) \subset D$, the group of diagonal matrices. Hence $\rho|\pi_1(W_2)$ factors through a representation $\rho_0 : H_1(W_2) \to D$ which, from our discussion above, determines ρ. Thus we are led to ask: which $\rho_0 : H_1(W_2) \to D$ can be so obtained? To answer this question, observe that if $\gamma \in \pi_1(W_2)$, then as $\rho(\gamma) \in D$ we have

$$\rho(\mu\gamma\mu^{-1}) = \rho(\mu)\rho(\gamma)\rho(\mu)^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rho(\gamma) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \rho(\gamma)^{-1} = \rho(\gamma^{-1}),$$

and therefore

$$\rho(\mu\gamma\mu^{-1}\gamma) = I \text{ for all } \gamma \in \pi_1(W_2).$$

Let $t \in Aut(W_2 \to W) = \mathbb{Z}/2$ be the generator and note that its action on $H_1(W_2) = \mathbb{Z} \oplus \operatorname{coker}(h^2 - 1_{H_1(T)})$ (Proposition 4.2) is given by:

$$t(\mu_2) = \mu_2, \qquad t(\alpha) = h(\alpha) \text{ for } \alpha \in \operatorname{coker}(h^2 - 1).$$

The identity $\rho(\mu\gamma\mu^{-1}\gamma) = I$ for $\gamma \in \pi_1(W_2)$ yields

$$\rho_0((1+t)H_1(W_2)) = \{I\}.$$

In other words, ρ_0 factors through $H_1(W_2)/(1+t)H_1(W_2)$. To determine this quotient, observe that $(1+t)(\mu_2) = 2\mu_2$ and as t acts as h on $\pi_1(T) \subset \pi_1(W_2)$, we see that $\text{coker}(h^2 - 1)/(h+1)\text{coker}(h^2-1) = \text{coker}(h+1)$. Hence

$$H_1(W_2)/(1+t)H_1(W_2) \cong \mathbb{Z}/2 \oplus \text{coker}(h+1)$$

where the $\mathbb{Z}/2$-factor is generated by the class of μ_2.

Conversely, if $\rho_1 : H_1(W_2)/(1+t)H_1(W_2) \to D$ is a given homomorphism which sends the class of μ_2 to $-I$ and we define ρ_0 to be the composition $H_1(W_2) \to H_1(W_2)/(1+t)H_1(W_2) \xrightarrow{\rho_1} D$, the identity $\rho_0(t(\bar{\gamma})) = \rho_0(-\bar{\gamma})$ is satisfied. It is then a simple matter to verify that we may define a representation $\rho : \pi_1(W) \to N$ by setting $\rho(\mu) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\rho|\pi_1(W_2)$ to be the composition $\pi_1(W_2) \to H_1(W_2) \xrightarrow{\rho_0} D$. The representation will be irreducible precisely when $\rho_0|\text{coker}(h^2 - 1_{H_1(T)})$ does not have image in $\{\pm I\}$, or equivalently, $\rho_1(\text{coker}(h + 1_{H_1(T)})) \not\subset \{\pm I\}$. Let

$$Z_h = Hom(\text{coker}(h + 1_{H_1(T)}), \{\pm I\}) \subset Hom(\text{coker}(h + 1_{H_1(T)}), D).$$

Lemma 4.4 *The set of $SL(2,\mathbb{C})$ conjugacy classes of irreducible representations $\pi_1(W) \to SL(2,\mathbb{C})$ corresponds bijectively with $(Hom(coker(h + 1_{H_1(T)}), D) \setminus Z_h)/(\psi = \psi^{-1})$. Moreover,*

- *if trace$(h) \neq -2$, then each such character is binary dihedral.*
- *if trace$(h) = -2$ but $h \neq -I$, then $X^{irr}(W)$ consists of $[\frac{n+2}{2}]$ curves of characters of representations with image in N, where n is the order of the torsion subgroup of $H_1(W)$.*
- *if $h = -I$, then $X^{irr}(W)$ is a 2-dimensional variety.*

Proof. The only part of the first statement which is left to verify is that an $SL(2,\mathbb{C})$-conjugacy between N-representations corresponds to replacing $\psi \in Hom(\text{coker}(h+1_{H_1(T)}), D)$ by either ψ or ψ^{-1}. We leave this as an elementary exercise.

To prove the second statement, note that if $\text{trace}(h) \neq -2$, then $\text{coker}(h + 1_{H_1(T)})$ is finite, and hence by our analysis above, each irreducible representation $\pi_1(W) \to N$ has a finite image and therefore is binary dihedral. On the other hand if $\text{trace}(h) = -2$, it is straightforward to verify the claimed results holds. $\diamondsuit$

We have proven the following proposition.

Proposition 4.5 *Let W be a torus bundle over S^1 with monodromy h where trace$(h) \neq -2$. Set $\zeta_h = \#Z_h \in \{1,2,4\}$. Then*

$$\#X^{irr}(W) = \frac{1}{2}(|2 + trace(h)| - \zeta_h).$$

Furthermore each such character is binary dihedral. $\diamondsuit$

Corollary 4.6 *Let W_n be the torus bundle over S^1 with monodromy h^n. If $|trace(h)| > 2$, then the number of irreducible characters of $\pi_1(W_n)$ tends to ∞ with n.*

Proof. Since $|\text{trace}(h)| > 2$, there is an eigenvalue λ of h with $|\lambda| > 1$. Then λ^n is an eigenvalue of h^n and therefore trace$(h^n) = \lambda^n + \lambda^{-n}$. Thus the number of conjugacy classes of irreducible representations $\pi_1(W_n) \to SL(2,\mathbb{C})$ is

$$\frac{1}{2}(|2 + \text{trace}(h^n)| - \zeta) = \begin{cases} 2\cosh(\frac{n\ln(|\lambda|)}{2}) - \frac{\zeta}{2} & \text{if } \lambda > 1 \text{ or } n \text{ is even} \\ \\ 2\sinh(\frac{n\ln(|\lambda|)}{2}) - \frac{\zeta}{2} & \text{if } \lambda < -1 \text{ and } n \text{ is odd.} \end{cases}$$

This proves the result. $\diamondsuit$

We can now verify that the hypotheses of Theorem A hold for hyperbolic punctured torus bundles over S^1.

Proof of Theorem B

We only need to show that for a hyperbolic punctured torus bundle M over S^1, all the conditions of Theorem A are satisfied. Obviously such a manifold is a knot exterior with Seifert surface (a fiber). By either [13] or [11], any hyperbolic punctured torus bundle over the circle is a small knot exterior, so condition (a) of the theorem holds. Condition (b) holds because of Proposition 4.5 and Corollary 4.6 if we observe that (i) the manifold W_n of these results is the manifold $M_n(\lambda_n)$ of the theorem, and (ii) the hyperbolicity of M is equivalent to the condition that the monodromy h of the torus bundle W_n satisfies $|\text{trace}(h)| > 2$. Finally by Proposition 4.3, any irreducible representation of $M_n(\lambda_n)$ sends $\pi_1(\partial M_n)$ to a group of order 4, which implies that condition (c) holds. $\diamondsuit$

5 Character varieties of punctured torus bundles over S^1

In this section we prove some general results concerning the $SL(2,\mathbb{C})$-character variety of a hyperbolic punctured torus bundle over S^1, M. Throughout, F will denote a fixed fibre of M and we shall assume that the monodromy of M is the identity on ∂F. Moreover we shall suppose that the base point of M lies in ∂F.

Let $h : \pi_1(F) \to \pi_1(F)$ be the monodromy-induced isomorphism. It is known that the condition that M be hyperbolic is equivalent to requiring that $h_* : H_1(F) \to H_1(F)$ be hyperbolic, i.e. $|\mathrm{trace}(h_*)| > 2$. We denote by $H : X(F) \to X(F)$ the algebraic equivalence determined by precomposition with h.

The *meridian* of M, denoted μ, is the trace under h of the base point of M. The *longitude*, denoted λ, is simply the boundary of F. Fix orientations for these curves. For the rest of the paper we shall also use μ and λ to denote the class of the meridian and longitude in either $H_1(\partial M) = \pi_1(\partial M), H_1(M)$ or $\pi_1(M)$. This gives us a canonical way to identify the slopes on ∂M with $\mathbb{Q} \cup \{\frac{1}{0}\}$ by associating the slope r with $\frac{p}{q}$ if $\pm(p\mu + q\lambda)$ is the pair of primitive homology classes in $H_1(\partial M)$ determined by r.

The fundamental group of M admits a presentation of the form

$$\pi_1(M) = < x, y, \mu \mid \mu x \mu^{-1} = h(x), \mu y \mu^{-1} = h(y) >$$

where x and y are free generators of $\pi_1(F)$ and μ corresponds to the meridian of M (after possibly alterring its orientation). Evidently the free group $\pi_1(F) = < x, y >$ is normal in $\pi_1(M)$. We can assume that x and y are chosen so that $\lambda = xyx^{-1}y^{-1}$.

Recall that a curve $Y_0 \subset X(F)$ is called *non-trivial* if it contains an irreducible character and τ_λ*-non-constant* if $\tau_\lambda | Y_0$ is non-constant (note $\lambda \in \pi_1(F)$). As in the proof of Proposition 2.6, it can be shown that if Y_0 is τ_λ-non-constant, then Y_0 is non-trivial.

Let $\hat{i} : X(M) \to X(F)$ be the regular map induced by inclusion. For each component X_0 of $X(M_n)$, the *restriction* of X_0 in $X(F)$ is the Zariski closure of its image under $\hat{i}$. Note that X_0 is τ_λ-non-constant if and only if its restriction to $X(F)$ is τ_λ-non-constant as well.

Our goal in this section is to establish the relationship between τ_λ-non-constant components of $X(M)$ and their restrictions in $X(F)$. Inspection of the presentation of $\pi_1(M)$ above shows that $H \circ \hat{i} = \hat{i}$, that is the image of $\hat{i}$ is contained in the fixed point set of H. Our first result shows that non-trivial curves which lie in this fixed point set arise as restrictions of curves from $X(M)$.

Proposition 5.1 *A τ_λ-non-constant curve Y_0 in $X(F)$ is the restriction of a τ_λ-non-constant curve component X_0 of $X(M)$ if and only if Y_0 is pointwise fixed by H.*

Proof. We have already observed that the image of $\hat{i}$ is contained in the fixed point set of H, so assume that Y_0 is a τ_λ-non-constant curve Y_0 in $X(F)$. As such curves are non-trivial, irreducible characters form a dense subset of Y_0. Let $\chi_\rho \in Y_0$ be an irreducible character. Since it is a fixed point of H, ρ and $\rho \circ h$ are conjugate representations of $\pi_1(F)$. Thus there is a matrix $A \in SL(2, \mathbb{C})$, uniquely determined up to sign, such that $A\rho(x)A^{-1} = \rho(h(x))$, $A\rho(y)A^{-1} = \rho(h(y))$, and $A\rho(xy)A^{-1} = \rho(h(xy))$. We can therefore extend ρ to two irreducible representations of $\pi_1(M)$ by setting $\rho(\mu) = A$ or $-A$. Hence $\hat{i}^{-1}(Y_0)$ is a subvariety of $X(M)$ of positive dimension. By Proposition 2.4, $\hat{i}^{-1}(Y_0)$ contains a non-trivial, τ_λ-non-constant curve component X_0. $\diamondsuit$

Recall from Remark 3.4 there is an action of $H^1(M;\mathbb{Z}/2) = Hom(\pi_1(M), \{\pm 1\})$ on $R(M)$ and $X(M)$. Let $\phi : \pi_1(M) \to \{\pm 1\}$ be the homomorphism determined by

$$\phi(x) = \phi(y) = 1, \phi(\mu) = -1$$

and recall the algebraic isomorphisms $\phi^* : R(M) \to R(M)$ and $\hat{\phi} : X(M) \to X(M)$. Since $\phi|\pi_1(F) \equiv I$, we see that

$$\hat{i}(\chi) = \hat{i}(\hat{\phi}(\chi)) \text{ for all } \chi \in X(M).$$

Thus from Proposition 3.3 we obtain:

Proposition 5.2 *Let X_0 be a τ_λ-non-constant curve component of $X(M)$. Then $\hat{i}|_{X_0}$ is a degree two or degree one map depending exactly on whether $\hat{\phi}(X_0) = X_0$ or $\hat{\phi}(X_0) \neq X_0$ respectively.* $\diamondsuit$

As an aid to the calculations of the next section, we are interested in finding conditions which guarantee that a given curve component of $X(M)$ is invariant under $\hat{\phi}$. To obtain such a condition, we must first examine the smoothness in $X(M)$ of the binary dihedral characters in $X(M(\lambda))$.

Recall that a point of a complex affine algebraic variety X is called a *simple point* if it is contained in a unique component X_0 of X and is a smooth point of X_0 [25].

Proposition 5.3 *Let M be a hyperbolic once-punctured torus bundle over S^1. Then every binary dihedral character in $X(M(\lambda)) \subset X(M)$ is a simple point of $X(M)$.*

Proof. There is an exact sequence

$$1 \to \pi_1(F) \to \pi_1(M) \to \mathbb{Z} \to 1$$

and there is a simple closed curve on ∂M whose associated class $\mu \in \pi_1(M)$ fits into a presentation

$$\pi_1(M(\lambda)) = < x, y, \mu \mid \mu x \mu^{-1} = x^p y^q, \mu y \mu^{-1} = x^r y^s, xy = yx >$$

where $h = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$ is a hyperbolic monodromy matrix for $M(\lambda)$, i.e $|p+s| > 2$.

Suppose that $\chi_\rho \in X(M(\lambda))$ is a binary dihedral character. From §4 we see that up to conjugation we may assume

$$\rho(\mu) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \rho(x) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \rho(y) = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$$

where α, β satisfy $\alpha^{p+1}\beta^q = 1$, $\alpha^r\beta^{s+1} = 1$ and either $\alpha \neq \pm 1$ or $\beta \neq \pm 1$. Appealing to [5, Theorem 3], we only need to show that $H^1(M(\lambda); sl(2,\mathbb{C})_\rho) = 0$, where $sl(2,\mathbb{C})_\rho$ is the $\pi_1(M(\lambda))$-module structure on the Lie algebra $sl(2,\mathbb{C})$ of $SL(2,\mathbb{C})$ induced by $\pi_1(M(\lambda)) \xrightarrow{\rho} SL(2,\mathbb{C}) \xrightarrow{Ad} sl(2,\mathbb{C})$. Equivalently, we need to show $H^1(\pi; sl(2,\mathbb{C})_\rho) = 0$ where $\pi = \pi_1(M(\lambda))$. Since ρ is non-abelian, it suffices to prove that the space of 1-cocycles, $Z^1(\pi; sl(2,\mathbb{C})_\rho)$, is 3-dimensional.

Any 1-cocycle $u \in Z^1(\pi; sl(2,\mathbb{C})_\rho)$ satisfies the *cocycle condition*

$$u(zz') = u(z) + Ad\rho(z)(u(z')) \qquad z, z' \in \pi$$

and so is determined by the trace zero matrices

$$u(x) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, u(y) = \begin{pmatrix} e & f \\ g & -e \end{pmatrix}, u(\mu) = \begin{pmatrix} u & v \\ w & -u \end{pmatrix}.$$

The cocycle condition implies that for each $z \in \pi$ and $n \in \mathbb{Z}$ we have

$$u(z^n) = \epsilon(n) \sum_{j=0}^{|n|-1} Ad\rho(z)^{(j+\frac{1-\epsilon(n)}{2})\epsilon(n)}(u(z)) \quad \text{where } \epsilon(n) = \begin{cases} 0 & \text{if } n = 0 \\ \frac{n}{|n|} & \text{otherwise.} \end{cases}$$

Hence the three relations $u(\mu x \mu^{-1}) = u(x^p y^q)$, $u(\mu y \mu^{-1}) = u(x^r y^s)$ and $u(xy) = u(yx)$ yield the following conditions:

- $$\begin{aligned}(1 - Ad\rho(\mu x \mu^{-1}))(u(\mu)) + Ad\rho(\mu)(u(x)) &= \epsilon(p)(\sum_{j=0}^{|p|-1} Ad\rho(x)^{(j+\frac{1-\epsilon(p)}{2})\epsilon(p)})(u(x)) \\ &+ \epsilon(q) Ad\rho(x)^p (\sum_{k=0}^{|q|-1} Ad\rho(y)^{(k+\frac{1-\epsilon(q)}{2})\epsilon(q)})(u(y)))\end{aligned}$$
- $$\begin{aligned}(1 - Ad\rho(\mu y \mu^{-1}))(u(\mu)) + Ad\rho(\mu)(u(y)) &= \epsilon(r)(\sum_{j=0}^{|r|-1} Ad\rho(x)^{(j+\frac{1-\epsilon(r)}{2})\epsilon(r)})(u(x)) \\ &+ \epsilon(s) Ad\rho(y)^r (\sum_{k=0}^{|s|-1} Ad\rho(y)^{(k+\frac{1-\epsilon(s)}{2})\epsilon(s)})(u(y)))\end{aligned}$$
- $u(x) + Ad\rho(x)(u(y)) = u(y) + Ad\rho(y)(u(x))$

Assuming that $\alpha \neq \pm 1$ and $\beta \neq \pm 1$ we obtain three matrix identities

- $\begin{pmatrix} -a & -c+(1-\alpha^{-2})v \\ -b+(1-\alpha^2)w & a \end{pmatrix} = \begin{pmatrix} pa+qe & \frac{(1-\alpha^{2p})}{(1-\alpha^2)}b + \frac{\alpha^{2p}(1-\beta^{2q})}{(1-\beta^2)}f \\ \frac{(1-\alpha^{-2p})}{(1-\alpha^{-2})}c + \frac{\alpha^{-2p}(1-\beta^{-2q})}{(1-\beta^{-2})}g & -(pa+qe) \end{pmatrix}$
- $\begin{pmatrix} -e & -g+(1-\beta^{-2})v \\ -f+(1-\beta^2)w & e \end{pmatrix} = \begin{pmatrix} ra+se & \frac{(1-\alpha^{2r})}{(1-\alpha^2)}b + \frac{\alpha^{2r}(1-\beta^{2s})}{(1-\beta^2)}f \\ \frac{(1-\alpha^{-2r})}{(1-\alpha^{-2})}c + \frac{\alpha^{-2r}(1-\beta^{-2s})}{(1-\beta^{-2})}g & -(ra+se) \end{pmatrix}$
- $\begin{pmatrix} a+e & b+\alpha^2 f \\ c+\alpha^{-2}g & -a-e \end{pmatrix} = \begin{pmatrix} a+e & f+\beta^2 b \\ g+\beta^{-2}c & -a-e \end{pmatrix}$

which can be converted into the following system of linear relations in nine variables $a, b, c, e, f, g, u, v, w$:

$$\begin{cases} (1) \quad (p+1)a + qe = 0 \\ (2) \quad \frac{(1-\alpha^{2p})}{(1-\alpha^2)}b + \frac{\alpha^{2p}(1-\beta^{2q})}{(1-\beta^2)}f + c - (1-\alpha^{-2})v = 0 \\ (3) \quad \frac{(1-\alpha^{-2p})}{(1-\alpha^{-2})}c + \frac{\alpha^{-2p}(1-\beta^{-2q})}{(1-\beta^{-2})}g + b - (1-\alpha^2)w = 0 \\ (4) \quad ra + (s+1)e = 0 \\ (5) \quad \frac{(1-\alpha^{2r})}{(1-\alpha^2)}b + \frac{\alpha^{2r}(1-\beta^{2s})}{(1-\beta^2)}f + g - (1-\beta^{-2})v = 0 \\ (6) \quad \frac{(1-\alpha^{-2r})}{(1-\alpha^{-2})}c + \frac{\alpha^{-2r}(1-\beta^{-2s})}{(1-\beta^{-2})}g + f - (1-\beta^2)w = 0 \\ (7) \quad (1-\beta^2)b + (\alpha^2-1)f = 0 \\ (8) \quad (1-\beta^{-2})c + (\alpha^{-2}-1)g = 0. \end{cases}$$

Since $|p+s| > 2$, equations (1) and (4) show $a = e = 0$. Next plugging into (2) the value for f determined by equation (7) yields

$$v = \frac{(1-\alpha^{2p}\beta^{2q})b}{(1-\alpha^2)(1-\alpha^{-2})} + \frac{c}{(1-\alpha^{-2})} = \frac{b}{1-\alpha^2} + \frac{c}{1-\alpha^{-2}}$$

since $\alpha^{p+1}\beta^q = 1$. Similarly (8) and (3) lead us to

$$w = \frac{b}{1-\alpha^2} + \frac{c}{1-\alpha^{-2}} = v.$$

Equations (5) and (6) add no new constraints, and so the solution space of this system of linear equations , i.e. the space $Z^1(\pi; sl(2,\mathbb{C})_\rho)$, is three dimensional.

A similar, though easier, argument deals with the cases $\alpha = \pm 1$, $\beta = \pm 1$. $\diamondsuit$

Corollary 5.4 *Let X_0 be a curve component in $X(M)$ which contains a binary dihedral character of $X(M(\lambda))$. Then $\hat{\phi}(X_0) = X_0$ and $\hat{i}|_{X_0}$ is a degree-two map to a curve $Y_0 \subset X(F)$.*

Proof. Since $\hat{\phi}$ is an isomorphism, $\hat{\phi}(X_0)$ is a curve component of $X(M_n)$. We already knew that if $\chi_\rho \in X(M(\lambda)) \subset X(M)$ is a binary dihedral character, then up to conjugation, $\rho(\mu) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\rho(x)$ and $\rho(y)$ are diagonal matrices. It is easy to see that $\hat{\phi}(\chi_\rho) = \chi_\rho$, so if $\chi_\rho \in X_0$, then $\chi_\rho \in \hat{\phi}(X_0)$ as well. Hence as χ_ρ is a simple point of $X(M)$ (Proposition 5.3), we have $\hat{\phi}(X_0) = X_0$. To complete the proof, we need only show that X_0 is τ_λ-non-constant (cf. Proposition 5.2). But this is a consequence of Proposition 2.5, as $\tau_\lambda(\chi_\rho) = 2$. $\diamondsuit$

6 Character varieties of finite cyclic covers of the figure-eight knot exterior

In this section we look more closely at the character varieties of the cyclic covers of the figure-eight knot exterior, which we shall denote by M. We shall continue to use the notation developed in the previous section.

The n-fold cyclic cover of M is known to be a hyperbolic punctured torus bundles whose fundamental group admits a presentation of the form

$$\pi_1(M_n) = < x, y, \mu_n \mid \mu_n x \mu_n^{-1} = h^n(x), \mu_n y \mu_n^{-1} = h^n(y) >$$

where h is the monodromy isomorphism given by

$$h(x) = xy, \quad h(y) = yxy,$$

and μ_n is the meridian of M_n. The longitude class of M_n, which we denote by λ_n, equals $xyx^{-1}y^{-1}$.

Let $H : X(F) \to X(F)$ be the algebraic equivalence determined by precomposition with h. Note that the M_n-monodromy isomorphism of $\pi_1(F)$ is simply h^n and the associated equivalence of $X(F)$ is H^n.

In what follows, we shall use the algebraic isomorphism

$$X(F) \to \mathbb{C}^3, \quad \chi \mapsto (\chi(x), \chi(y), \chi(xy))$$

to identify $X(F)$ with $\mathbb{C}^3$. Using the trace identity

$$trace(AB) + trace(AB^{-1}) = trace(A)trace(B)$$

one can deduce that the map $H : X(F) \to X(F)$ is given by:

$$H(a, b, c) = (c, bc - a, (bc - a)c - b),$$

and the trace function $\tau_{\lambda_n} : X(F) \to \mathbb{C}$ equals:

$$\tau_{\lambda_n}(a,b,c) = a^2 + b^2 + c^2 - abc - 2.$$

Recall from Remark 3.4 there is an action of $H^1(M_n;\mathbb{Z}/2) = Hom(\pi_1(M_n),\{\pm 1\})$ on $R(M_n)$ and $X(M_n)$. A simple argument based on first principles shows that this action preserves the type of a curve component of $X(M)$ (or see the method of proof of [4, Lemma 5.4]).

Let $\phi_n : \pi_1(M_n) \to \{\pm 1\}$ denote the homomorphism determined by

$$\phi_n(x) = \phi_n(y) = 1, \phi_n(\mu_n) = -1.$$

The restriction map $X(M_n) \to X(F)$ will be denoted by i_n. Evidently

$$\hat{i}_n \circ \hat{\phi}_n = \hat{i}_n : X(M_n) \to X(M_n).$$

We are now ready to determine the number and the types of nontrivial curve components in $X(M_n)$ for $n = 2, 3$.

Proposition 6.1 *$X(M_2)$ has exactly two curve components of type (i). It also has exactly two nontrivial curve components of type (iii) whose associated boundary slopes are 2 and -2 respectively.*

Proof. We first find all nontrivial curve components of $X(M_2)$ by applying Proposition 5.1. This amounts to finding the curve components in the fixed point set of $H^2 : X(F) \to X(F)$. A simple calculation reveals that for $(a,b,c) \in \mathbb{C}^3 = X(F)$, $H^2(a,b,c) = (a,b,c)$ if and only if

$$\begin{cases} (bc-a)c - b = a \\ (bc-a)^2 c - (bc-a)b - c = b \\ (bc-a)^3(c^2-2) - 2(bc-a)^2 a + (bc-a)(b-c) + a = c \end{cases}$$

or equivalently

$$\begin{cases} (1)\ (c+1)(bc-a-b) = 0 \\ (2)\ (ac+b-bc^2+c)(a-1-bc) = 0 \\ (3)\ (ac-1+b-bc^2)(a^2c - 2abc^2 + ab + a + b^2c^3 - c - b^2c - bc) = 0. \end{cases}$$

From Equation (1) we deduce that either $c = -1$ or $a = -b + bc$. When $c = -1$, Equations (2) and (3) become $(1+a)(a+b-1) = 0$ and $(1+a)(ab - b + a^2 - a - 1) = 0$ respectively. This produces exactly one curve:

$$Y_1 = \{(-1, b, -1) \mid b \in \mathbb{C}\} \subset \mathbb{C}^3 = X(F).$$

When $a = -b + bc$, Equations (2) and (3) become $(1+b)(bc-b-c) = 0$ and $(1+b)(bc-c-b)(bc-b+1) = 0$, for which there are exactly two solution curves:

$$Y_2 = \{(a, -1, 1-a) \mid a \in \mathbb{C}\},$$

$$Y_3 = \{(a, \frac{a}{a-1}, a) \mid a \in \mathbb{C} \setminus \{1\}\}.$$

One can easily check that on each of the three curves the trace function τ_{λ_2} is non-constant. Thus by Proposition 5.1, all three are restrictions of nontrivial curve components of $X(M_2)$. Our next task will be to show that both Y_1 and Y_2 are the restrictions of unique curves X_1 and X_2 in $X(M_2)$, while there are exactly two curves $X_3, X_4 \subset X(M_2)$ which restrict to Y_3.

According to Proposition 4.5, there are exactly four binary dihedral characters in $X(M_2(\lambda_2)) \subset X(M_2)$. An explicit calculation based on the discussion in §4 shows that the images of these characters under the map $\hat{i}_2$ are the following points in $X(F) = \mathbb{C}^3$:

$$(-1, 2, -1), (-1, -1, -1), (-1, -1, 2), \text{ and } (2, -1, -1).$$

One can easily check that the first two points are contained in $Y_1 \setminus (Y_2 \cup Y_3)$ while the last two are contained in $Y_2 \setminus (Y_1 \cup Y_3)$. Therefore it follows from Proposition 5.2 and Corollary 5.4 that there is exactly one curve component X_1 of $X(M_2)$ which restricts to Y_1, and one curve component X_2 which restricts to Y_2.

Now we show that there are two type (i) curve components of $X(M_2)$ which restrict to Y_3. By the results of §5 it suffices to prove that there is a curve component X_3 of $X(M_2)$ which restricts to Y_3 and for which $\hat{i}_2|_{X_3}$ is a degree one map.

Recall that $X(M)$ is known to have only one nontrivial curve component X_0 (see eg. [6]), which is a type (i) curve. It follows that $\hat{\phi}_1(X_0) = X_0$ and hence $\hat{i}_1|_{X_0}$ is a degree-two map onto its image. The covering map $p_2 : M_2 \rightarrow M$ induces a restriction $\hat{p}_2 : X(M) \rightarrow X(M_2)$. If $X_3 \subset X(M_2)$ is the restriction of X_0, then X_3 is a type (i) curve component of $X(M_2)$ (Lemma 3.8). The same lemma implies that X_3 restricts to Y_3. Since $\pi_1(M_2)$ is the unique index 2 subgroup of $\pi_1(M)$, $\phi_1|\pi_1(M_2)$ is trivial. Thus $\hat{p}_2 \circ \hat{\phi}_1 = \hat{p}_2$ and therefore $\hat{p}_2|_{X_0}$ is a degree two map to X_3. Hence from the commutative diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{\hat{i}_1} & X(F) \\ \hat{p}_2 \downarrow & & \downarrow = \\ X_3 & \xrightarrow{\hat{i}_2} & X(F). \end{array}$$

we deduce that $\hat{i}_2|_{X_3}$ is of degree one.

Finally we shall show that X_1 and X_2 are type (iii) curve components of $X(M_2)$ whose associated slopes are 2 and -2 respectively. We shall assume that the reader is familiar with

[13] and the algorithm described there which calculates the boundary slopes of punctured torus bundles over the circle. Accordingly we note that essential surfaces in such manifolds correspond to certain “minimal edge paths” in the diagram of $PSL_2(\mathbb{Z})$. From the minimal edge path associated to a given essential surface S in M_2, one can read off a sequence elements of $\pi_1(S)$, well-defined up to conjugation in $\pi_1(M_2)$, and expressed in terms of x and y.

According to either [13] (or [11]), the set of boundary slopes of M_2 are $2, 0, -2$ and the components of any essential surface in M_2 of slope 0 are isotopic to punctured torus fibres, i.e. 0 is not a strict boundary slope. If we observe that an irreducible character in $X(F)$ of a representation conjugate into N (§4) corresponds to (a, b, c) where one of a, b or c is 0, then the generic character in Y_1, Y_2, Y_3 is not the character of such a representation. It then follows from [3, Proposition 4.7 (2)] and the method of proof of [4, Proposition 4.2] that the 0 slope is not associated to an ideal point of any non-trivial curve in $X(M_2)$. We also note that the method of [13] shows that any essential surface with boundary slope -2 contains a loop which represents y, at least up to conjugation. Similarly any essential surface with boundary slope 2 contains a loop which represents x, up to conjugation.

Now consider an ideal point x_1 of $\tilde{X}_1$. Since M_2 is small, x_1 is associated to some boundary slope of M_2 which, as we have just noted, must be either 2 or -2. In particular τ_{λ_2} has a pole at x_1. Hence the image of x_1 in $\tilde{Y}_1$ under the restriction induced map $\tilde{X}_1 \to \tilde{Y}_1$, call it y_1, is also ideal. From the calculations above, Y_1 is a complex line and thus $\tilde{Y}_1 \cong \mathbb{C}P^1$ has a unique ideal point, namely y_1. These calculations also show that $\tau_x(y_1) = \tau_{xy}(y_1) = -1$ while τ_y has a pole there. In particular, the y cannot conjugate into the fundamental group of any essential surface associated to x_1. Thus from the previous paragraph we see that x_1 must be associated to the slope 2. Since x_1 was an arbitrary ideal point of X_1 it follows that $f_2|X_1$ is constant. Hence X_1 is a type (iii) curve associated to the slope 2.

Similarly one can show that if X_2 is a type (iii) curve whose associated boundary slope is -2. $\diamondsuit$

Proof of Theorem C

By Proposition 6.1, M_2 has two type (iii) curves X_1 and X_2 whose associated boundary slopes are 2 and -2 respectively. By Lemma 3.8, the restriction of X_i to M_{2^k}, call it Y_{ki}, is still a type (iii) curve in $X(M_{2^k})$. The associated boundary slopes are $1/2^{k-2}$ and $-1/2^{k-2}$ respectively. The proof is completed by noting that the distance between the slopes $1/2^{k-2}$ and $-1/2^{k-2}$ is 2^{k-1}. $\diamondsuit$

Proposition 6.2 $X(M_3)$ *has exactly four type* (*i*) *curve components and six nontrivial type* (*iii*) *curve components. The boundary slopes associated to the latter curves are each* *the meridian slope* μ_3*.*

Proof. The approach is similar to that we used in the proof of Proposition 6.1, though slightly more involved owing to the increased complexity of H^3.

The fixed point set of $H^3 : X(F) \rightarrow X(F)$ is given by the solutions to the following three equations in a, b, c:

(4) $0 = (-a + bc)(-b - c - ac + bc^2)(-b + c - ac + bc^2)$

(5) $0 = (ac + b - bc^2)(a^2c - a + ab - c + bc - b^2c - 2abc^2 + b^2c^3)(a^2c + a + ab - c - bc - b^2c - 2abc^2 + b^2c^3)$

(6) $0 = (a^2c + ab - c - b^2c - 2abc^2 + b^2c^3)(ac - bc^2 + 3ab^2c^4 + bc^3 - 4ab^2c^2 - a + b + 2a^2bc + ab^2 + a^3c^2 + 2b^3c^3 - b^3c - ac^2 - 3a^2bc^3 - b^3c^5)(-ac + bc^2 + 3ab^2c^4 + bc^3 - 4ab^2c^2 - a - b + 2a^2bc + ab^2 + a^3c^2 + 2b^3c^3 - b^3c - ac^2 - 3a^2bc^3 - b^3c^5)$.

It is straightforward to verify that

$$Y_1 = \{(a, 0, 0) \mid a \in \mathbb{C}\}$$

is a solution curve of these equations. Next observe that the only other solution curves for which $b = 0$, respectively $c = 0$, are

$$Y_2 = \{(0, 0, c) \mid c \in \mathbb{C}\},$$

respectively

$$Y_3 = \{(0, b, 0) \mid b \in \mathbb{C}\}.$$

Thus we shall assume below that neither b nor c is identically zero.

From Equation (4) we see that either $a = bc$ or $0 = -b - c - ac + bc^2$ or $0 = -b + c - ac + bc^2$. The first case cannot arise, for if it did, Equation (5) becomes $bc^2 = 0$, contradicting the assumption we made at the end of the last paragraph. In the second case we have $a = -\frac{b}{c} - 1 + bc$ and so Equations (5) and (6) become

$(-b - c + bc)(b + c + bc) = 0,$

$b(c^2 - 1)(-b - c + bc)(b + c + bc) = 0.$

This produces the two curves:

$$Y_4 = \{(a, \frac{a}{a-1}, a) \mid a \neq 1\}$$

and

$$Y_5 = \{(a, \frac{-a}{a-1}, -a) \mid a \neq 1\}.$$

In the last case we have $a = -\frac{b}{c} + 1 + bc$ which gives rise to

$$Y_6 = \{(a, \frac{a}{a+1}, a) \mid a \neq -1\}$$

and

$$Y_7 = \{(a, \frac{-a}{a+1}, -a) \mid a \neq -1\}.$$

One can easily check that on each Y_i, the trace function τ_{λ_3} is non-constant and therefore by Proposition 5.1 we may choose a non-trivial τ_{λ_3}-non-constant component $X_i \subset X(M_3)$ which restricts to Y_i, $i = 1, 2, \ldots, 7$.

In order to see that X_1 is a type (iii) curve whose associated boundary slope is μ_3, we first observe that Y_1 is the set of characters of the representations $\rho_a : \pi_1(F) \to N$ $(a \in \mathbb{C})$ which are defined by

$$\rho_a(x) = \begin{pmatrix} \frac{a+\sqrt{a^2-4}}{2} & 0 \\ 0 & \frac{a-\sqrt{a^2-4}}{2} \end{pmatrix}, \quad \rho_a(y) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that ρ_a is irreducible as long as $a \neq \pm 2$. We know that ρ_a extends to $\pi_1(M_3)$ in such a way that $\rho_a(h^3(x)) = \rho_a(\mu_3)\rho_a(x)\rho_a(\mu_3)^{-1}$ and $\rho_a(h^3(y)) = \rho_a(\mu_3)\rho_a(y)\rho_a(\mu_3)^{-1}$. Now

$$h^3(x) = xy^2xy^2xyxy^2xy, \quad h^3(y) = yxyxy^2xyxy^2xy^2xyxy^2xy$$

and a direct calculation now shows that $\rho_a(h^3(x)) = \rho_a(x)$ and $\rho_a(h^3(y)) = \rho_a(y)$. Thus for $a \neq \pm 2$ we must have $\rho_a(\mu_3) = \pm I$. It follows that $\tau_{\mu_3}|X_1$ is identically 2 or -2, which is what we set out to prove.

We can also deduce that X_1 cannot be invariant under $\hat{\phi}_3$, and therefore there are precisely two curve components of $X(M_3)$ which restrict to Y_1 (cf. Proposition 5.2). For

$$\tau_{\mu_3} \circ \hat{\phi}_3|X_1 = \hat{\phi}_3(\mu_3)\tau_{\mu_3}|X_1 = -\tau_{\mu_3}|X_1 \neq \tau_{\mu_3}|X_1,$$

since $\tau_{\mu_3}|X_1$ is constantly 2 or -2.

A similar argument can be used to see that X_2 and X_3 are also type (iii) curves associated to the slope μ_3 and that there are precisely two curves in $X(M_3)$ which restrict to each of Y_2 and Y_3.

Next we consider the curves X_4, X_5, X_6, X_7. Let $X_0 \subset X(M)$ be the unique non-trivial curve, which we remind the reader is of type (i). Let Y_0 be the restriction of X_0 in $X(F)$ and Z_0 be the image of X_0 in $X(M_3)$ under the restriction $\hat{p}_3 : X(M) \to X(M_3)$ induced by a covering map $p_3 : M_3 \to M$. By Lemma 3.8, Z_0 is a curve of type (i) Furthermore, since $\hat{i}_3 \circ \hat{p}_3 = \hat{i}_1$, Y_0 is the restriction of Z_0 to $X(F)$. It follows that $Y_0 = Y_i$ for some $i = 4, 5, 6, 7$. Without loss of generality we take $i = 4$ and $X_4 = Z_0$.

Observation reveals that (a) the curves Y_4, Y_5, Y_6, Y_7 form an $H^1(F;\mathbb{Z}/2)$-orbit of curves in $X(F)$ and (b) each element of $H^1(F;\mathbb{Z}/2) = Hom(\pi_1(F), \{\pm 1\})$ extends to an element of $H^1(M_3;\mathbb{Z}/2)$. Hence X_4, X_5, X_6, X_7 form part of a $H^1(M_3;\mathbb{Z}/2)$-orbit of curves in $X(M_3)$. In particular they are all curves of type (i). Since the boundary slopes associated to the ideal points of X_0 are 4 and -4, it follows that those associated to the ideal points of X_4, X_5, X_6, X_7 are $4/3$ and $-4/3$.

Finally observe that since $\hat{\phi}_1(X_0) = X_0$ and $\hat{p}_3 \circ \hat{\phi}_1 = \hat{\phi}_3 \circ \hat{p}_3$, we have $\hat{\phi}_3(X_4) = \hat{\phi}_3(\hat{p}_3(X_0)) = \hat{p}_3(\hat{\phi}_1(X_0)) = \hat{p}_3(X_0) = X_4$. Then by Proposition 5.2, X_4 is the unique curve in $X(M_3)$ which restricts to Y_4. Since X_5, X_6, and X_7 all lie in the $H^1(M_3;\mathbb{Z}/2)$-orbit of X_4, they are the unique curves in $X(M_3)$ which restrict to Y_5, Y_6 and Y_7 respectively.

Alternately, a direct calculation shows that $X(M_3)$ contains exactly eight binary dihedral characters and their images in $X(F) = \mathbb{C}^3$ under $\hat{\phi}_3$ are:

$$\begin{array}{l}
(2\cos\pi/5, 2\cos 3\pi/5, 2\cos 4\pi/5)\\
(2\cos\pi/5, 2\cos 8\pi/5, 2\cos 9\pi/5)\\
(2\cos 2\pi/5, 2\cos\pi/5, 2\cos 3\pi/5)\\
(2\cos 2\pi/5, 2\cos 6\pi/5, 2\cos 8\pi/5)\\
(2\cos 3\pi/5, 2\cos 4\pi/5, 2\cos 7\pi/5)\\
(2\cos 3\pi/5, 2\cos 9\pi/5, 2\cos 2\pi/5)\\
(2\cos 4\pi/5, 2\cos 2\pi/5, 2\cos 6\pi/5)\\
(2\cos 4\pi/5, 2\cos 7\pi/5, 2\cos 1\pi/5).
\end{array}$$

One can check that the 4th point and the 7th point are contained in Y_4 and no other Y_j; the 3rd point and the 8th point are contained in Y_5 only; the 2nd point and the 5th point are contained in Y_6 only; the 1st point and the 6th point are contained in Y_7 only. Therefore by Corollary 5.4 and Proposition 5.2, there is exactly one curve component X_i of $X(M_3)$ which restricts to Y_i for each of $i = 4, 5, 6, 7$. $\diamondsuit$

7 Discrete faithful characters and norm curve components

In this section we shall describe a method which produces hyperbolic knot exteriors with large numbers of norm curve components in their character varieties. Recall that for a knot exterior M, each element $\epsilon \in H^1(M;\mathbb{Z}/2) = Hom(\pi_1(M), \mathbb{Z}/2)$ induces isomorphisms ϵ^* of $R(M)$ and $\hat{\epsilon}$ of $X(M)$ (see Remark 3.4). A simple argument based on first principles shows that norm curve components of $X(M)$ are preserved by this action (or see [4, Lemma 5.4]). Also recall that for a hyperbolic knot exterior M there are precisely $2|H^1(M;\mathbb{Z}/2)|$ characters of discrete faithful representations of $\pi_1(M)$ into $SL(2,\mathbb{C})$. In fact if χ_0 is such a character, then the set of all such characters in $X(M)$ is given by

$$\{\hat{\epsilon}(\chi_0), \hat{\epsilon}(\bar{\chi}_0) \mid \epsilon \in H^1(M;\mathbb{Z}/2)\}$$

where $\bar{\chi}$ denotes the complex conjugate of χ_0.

Proposition 7.1 *Let M be a hyperbolic knot exterior in S^3 and $p : M_n \to M$ be the n-fold cyclic cover. Then $X(M_n)$ contains at least $\frac{1}{2}\#H_1(M_n;\mathbb{Z}/2)$ norm curve components, each of which contains a discrete faithful character.*

Proof. Let $X_0 \subset X(M)$ be a norm curve component containing a discrete faithful character χ_{ρ_1}. Let $Y_0 \subset X(M_n)$ be the restriction of X_0 on $X(M_n)$. By Proposition 3.1, $Y_0 = \hat{p}(X_0)$ where $\hat{p} : X(M) \to X(M_n)$ is the regular map induced by the covering. Note that Y_0 is a norm curve component of $X(M_n)$ which contains the character of the discrete faithful representation $\rho_n = \rho_1|\pi_1(M_n)$.

Denote by ϵ_1 the unique non-zero element of $H^1(M;\mathbb{Z}/2)$ and define

$$\epsilon_n = \epsilon_1|\pi_1(M_n) \in H^1(M_n;\mathbb{Z}/2).$$

Suppose that $\epsilon \in H^1(M_n,\mathbb{Z}/2) \setminus \{0, \epsilon_n\}$ and consider the isomorphism $\hat{\epsilon} : X(M_n) \to X(M_n)$ induced by ϵ. We remarked above that $\hat{\epsilon}(Y_0)$ is a norm curve component. It contains the discrete faithful character $\epsilon\chi_{\rho_n}$.

<u>Claim</u> $\hat{\epsilon}(Y_0) \neq Y_0$.

<u>Proof of the Claim</u> Suppose otherwise and observe that as $\hat{p}(X_0) = Y_0$ (Proposition 3.1), there is a point $\chi_\rho \in X_0$ such that $\hat{p}(\chi_\rho) = \hat{\epsilon}(\chi_{\rho_n})$. Since $\rho(\pi_1(M)) \subset SL(2,\mathbb{C})$ contains a finite index subgroup which is discrete in $SL(2,\mathbb{C})$, $\rho(\pi_1(M))$ is also discrete in $SL(2,\mathbb{C})$. Note as well that if $\rho(\delta) = I$ for some nontrivial element $\delta \in \pi_1(M)$, then $\rho(\delta^n) = I$. But this is impossible because $\delta^n \in \pi_1(M_n) \setminus \{1\}$, since $\pi_1(M)$ is torsion free, and $\rho|\pi_1(M_n)$ is faithful. Hence we see that ρ is a discrete faithful representation of $\pi_1(M)$. It follows that $\chi_\rho \in \{\chi_{\rho_1}, \epsilon_1\chi_{\rho_1}, \bar{\chi}_{\rho_1}, \epsilon_1\bar{\chi}_{\rho_1}\}$ and therefore

$$\epsilon\chi_{\rho_n} \in \{\chi_{\rho_n}, \epsilon_n\chi_{\rho_n}, \bar{\chi}_{\rho_n}, \epsilon_n\bar{\chi}_{\rho_n}\}.$$

If $\epsilon\chi_{\rho_n} \in \{\bar{\chi}_{\rho_n}, \epsilon_n\bar{\chi}_{\rho_n}\}$, then there is a finite index subgroup Γ of $\pi_1(M)$ contained in $\ker(\epsilon) \cap \ker(\epsilon_n)$ such that $\chi_{\rho_1}|\Gamma = \bar{\chi}_{\rho_1}|\Gamma$, something which is impossible since Γ is a discrete subgroup of cofinite volume. On the other hand $\epsilon\chi_{\rho_n} \notin \{\chi_{\rho_n}, \epsilon_n\chi_{\rho_n}\}$ since $\epsilon \notin \{0, \epsilon_n\}$ and χ_{ρ_n} does not take on the value 0. This contradiction completes the proof of the claim.

Now suppose that $\epsilon, \epsilon' \in H^1(M_n;\mathbb{Z}/2)$ and $\hat{\epsilon}(Y_0) = \hat{\epsilon}'(Y_0)$. Then $\hat{\epsilon}'(\hat{\epsilon}(Y_0)) = Y_0$, which implies that $\epsilon' + \epsilon \in \{0, \epsilon_n\}$. It follows that the orbit of Y_0 under the action of $H^1(M_n;\mathbb{Z}/2)$ has at least $\frac{1}{2}\#H^1(M_n;\mathbb{Z}/2)$ elements. $\diamondsuit$

Thus in order to construct knot exteriors with large numbers of norm curve components, we need to find hyperbolic knot exteriors in S^3 having cyclic covers with large rank in $\mathbb{Z}/2$-homology. The Alexander polynomial can be used to find such knots, for it is known that the first Betti number of the cyclic cover M_n of the exterior M of a knot $K \subset S^3$ is equal to one plus the number of roots of the Alexander polynomials of K which are n-th roots of unity. Now any polynomial $A(t)$ having integer coefficients and even degree which satisfies the two conditions $A(t^{-1}) = t^n A(t)$, some $n \in \mathbb{Z}$, and $A(1) = \pm 1$, can be realized as the Alexander polynomial of a knot $K \subset S^3$ [18]. In fact it can be realized as the Alexander polynomial for infinitely many distinct hyperbolic knots [9].

Proof of Theorem D

For instance, take $A(t) = t^k - t^{k-1} + t^{k-2} - ... + t^2 - t + 1 = (t^{k+1}+1)/(t+1)$, where k is any even integer larger than 1, and realize it as the Alexander polynomial of a hyperbolic knot $K \subset S^3$. By our remarks above, the $2(k+1)$-fold cover of the exterior of such a knot has $\mathbb{Z}/2$-rank at least $k+1$. Hence by Proposition 7.1, its character variety contains at least 2^k curve components and each of them contains a discrete faithful character. $\diamondsuit$

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